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Improved collocation methods for thermal regenerator simulations

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Abstract—Robust and accurate collocation schemes are offered for the series solution of the classical Nusselt integral equations which describe the steady state temperature behaviour of a thermal regenerator. An explanation is provided of why the collocation and Galerkin series expansion methods are robust for the long regenerator problem, whereas the quadrature methods are not. A novel, economical method is provided for series expansion techniques for the non-symmetric case.

1. INTRODUCTION

THIS PAPER is concerned with the numerical solution of the integral equation

$$F(\Lambda-\xi)+e^{-\Pi}F(\xi)+\int_0^{\xi}K(\xi-\varepsilon)F(\varepsilon)\,d\varepsilon=1 \quad (1)$$

and the pair of simultaneous integral equations

$$F'(\xi')=e^{-\Pi'}F(\xi')+\int_{\xi'}^{\Lambda'}K'(\varepsilon-\xi')F(\varepsilon)\,d\varepsilon \quad (2)$$

$$(1-F(\xi))=e^{-\Pi}(1-F'(\xi))+\int_0^{\xi}K(\xi-\varepsilon)(1-F'(\varepsilon))\,d\varepsilon. \quad (3)$$

The *kernel* of these integral equations is defined by

$$K(x)=\frac{-iJ_1(2i(x\Pi)^{1/2})}{(x\Pi)^{1/2}}\Pi e^{-x-\Pi} \quad (4)$$

where $iJ_1(iy)$ is a real valued function with complex argument iy, where $i^2=-1$ and J_1 is the Bessel function of the first type and of first order. Iliffe [1] offered a similar set of integral equations, developed from those devised by Nusselt [2, 3].

These equations describe the relationship between the spatial solid temperature distribution, $F(\xi)$, for $0\leqslant\xi\leqslant\Lambda$, at the end of the *hot period* of thermal regenerator operation and the corresponding distribution $F'(\xi')$, for $0\leqslant\xi'\leqslant\Lambda'$, for the end of the *cold period.*

On the dimensionless scales introduced by Hausen [4] in 1929, the length of the regenerator in the hot period is Λ and in the cold period, Λ'. These parameters were called the *reduced length* by Hausen. The dimensionless duration of each period is given by the *reduced period*, Π, for the hot period and Π' for the cold.

Equations (2) and (3) deal with the general non-symmetric case where $\Lambda\neq\Lambda'$ and/or $\Pi\neq\Pi'$. Note that equation (2) assumes that gas flows through the packing in the cold period, entering the regenerator at $\xi'=\Lambda'$ and departing at position $\xi'=0$.†

Equation (1) exploits the symmetry of the case where $\Lambda=\Lambda'$ and $\Pi=\Pi'$ for which it can be shown that

$$F'(\Lambda-\xi)+F(\xi)=1. \quad (5)$$

These integral equations are Volterra equations of the second kind. Baker [5] recalls that the numerical techniques most favoured for such equations fall into two classes. In the *series expansion* methods, with which we are concerned here, we seek to represent $F(\xi)$ and $F'(\xi')$ by

$$F(\xi)=\sum_{j=0}^{n}\alpha_j\phi_j(\xi) \quad (6)$$

and

$$F'(\xi')=\sum_{j=0}^{n}\beta_j\phi_j(\xi'). \quad (7)$$

In the *quadrature* methods, such as that of Iliffe [1], the integral in equation (1), for example,

$$\int_0^{\xi}K(\xi-\varepsilon)F(\varepsilon)\,d\varepsilon$$

is approximated by one of the Newton–Coles methods (see Baker [5] or Delves and Mohamed [6]) and the solution of the integral equations consists of the vectors $\mathbf{F}$ and $\mathbf{F}'$ where $\mathbf{F}=[F_0, F_1, F_2, \ldots, F_n]^T$ and $\mathbf{F}'=[F'_0, F'_1, F'_2, \ldots, F'_n]^T$ and the F'_ks are temperatures at the entrance and exit to the regenerator, and at equally spaced, intermediate positions. On the

† In Iliffe's paper [1], for example, the equation is written with the gas entering at $\xi'=0$ and leaving at $\xi'=\Lambda'$.

NOMENCLATURE

F, $\mathbf{F}$	dimensionless temperature distribution at the end of the hot period
K	kernel of the integral equations defined by formula (4)
R	residual function defined by equation (12)
T_k	Chebyshev polynomial of degree k.

Greek symbols

α_i, β_i	coefficients in expansions (6) and (7)
Λ	reduced length of the regenerator
ξ	dimensionless distance from the hot gas entrance
Π	reduced period of regenerator operation
ϕ_j	function of degree j
Ω	function defined by equation (11).

Superscripts

$'$	refers to the end of the cold period
(p)	refers to the pth cycle.

other hand, the series expansion solution consists of vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ where $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n]^T$ and $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \ldots, \beta_n]^T$.

The series expansions (6) and (7) embody a set of *linearly independent* functions $\{\phi_i(\xi) | j = 0, 1, 2, \ldots, n\}$. Without loss of generality, substitution of expansion (6) into equation (1) yields

$$\sum_{j=0}^{n} \alpha_j \left\{ \phi_j(\Lambda - \xi) + e^{-\Pi} \phi_j(\xi) + \int_0^{\xi} K(\xi - \varepsilon) \phi_j(\varepsilon)\, d\varepsilon \right\} = 1. \quad (8)$$

In the collocation method, we apply equation (8) at $n+1$ *distinct* but not necessarily equally spaced positions $\{\xi_i | i = 0, 1, 2, \ldots, n\}$ yielding the matrix equation

$$A\boldsymbol{\alpha} = \mathbf{e}. \quad (9)$$

The matrix $A = [a_{i,j}]$ is defined by

$$a_{i,j} = \phi_j(\Lambda - \xi_i) + e^{-\Pi} \phi_j(\xi_i) + \int_0^{\xi_i} K(\xi_i - \varepsilon) \phi_j(\varepsilon)\, d\varepsilon$$

and the vector $\mathbf{e} = [1, 1, \ldots, 1]^T$.

In this paper, we suggest another set of functions $\{\phi_j(\xi) | j = 0, 1, \ldots, n\}$ to those proposed previously and introduce a novel method for handling the general case represented by equations (2) and (3). In so doing, we introduce significant economies which can be achieved in the solution of the *non-symmetric* problem. We also address two other issues. Why does the Iliffe [1] method break down, in the manner described by Willmott and Thomas [7] whereas this collocation method, and other series expansion techniques, do not, once certain precautions are taken in the calculation of the matrix elements $a_{i,j}$? How can economies be made in the number of terms in the series expansion required to represent regenerator behaviour accurately? In addressing this problem, we include a look at the choice of the possible data points which might be used.

2. THERMAL REGENERATOR OPERATION AND ITS MODELLING

Waste thermal energy is extracted from a hot gas as it passes through a relatively cool, porous packing with thermal capacitance for the duration of the *hot period*. This heat is recovered by a cold gas which flows through the same packing in the contra-flow direction during the *cold period*. This alternate heating and cooling of the packing imposes forced oscillations of temperature upon it, which become periodic after a sufficient number of cycles. A cycle consists of a cold period followed by a hot period. This periodic state is known as *cyclic equilibrium* and it is this condition to which equations (1)–(3) refer.

Equation (2), therefore, represents the cooling of the packing at cyclic equilibrium from an initial temperature distribution, $F(\xi')$ to a distribution at the end of the cold period equal to $F'(\xi')$. Similarly, equation (3) represents a hot period of operation. For the symmetric case we need only consider a cold period and use the symmetry conditions which apply at cyclic equilibrium, as contained in equation (5), thus yielding equation (1).

These equations are based on the linear model in which it is assumed that, in a given period of operation, the thermophysical properties of the gas and the solid packing, together with the heat transfer coefficients are constant and do not vary spatially or chronologically. Equally, it is assumed that the inlet temperature and flow rate of the gas do not vary with time within a period, hot or cold. Clearly, for the non-symmetric case, we allow the descriptive parameters in the hot period to be different from those in the cold period.

The linear model includes further a dimensionless temperature so that, on this scale, the hot gas enters at a temperature equal to 1 and the cold gas enters at temperature 0. This means that the temperature distributions $F(\xi)$ and $F'(\xi')$ all lie on a [0, 1] scale, a factor we exploit in our choice of linearly independent functions in the expansions (6) and (7).

A further discussion of the linear model together with the dimensionless parameters Λ, Λ', Π and Π' is

offered by Schmidt and Willmott [8] and also Hausen [9].

3. CHOICE OF LINEARLY INDEPENDENT FUNCTIONS

Nahavandi and Weinstein [10] first introduced the method of collocation to equations (1)–(3). They employed simple functions $\phi_k(\xi) = \xi^k$. However, as Willmott and Duggan [11] and subsequently Baclic [12] pointed out, very large matrix elements

$$a_{i,j} = (\Lambda - \xi_i)^j + e^{-\Pi}\xi_i^j + \int_0^{\xi_i} K(\xi_i - \varepsilon)\varepsilon^j \, d\varepsilon$$

can arise for large values of reduced length, Λ, for modestly large values of n, the maximum degree of the polynomial ξ^j used. These large elements can result in the process of solving the linear equation (9) breaking down, thereby limiting the possible range of application of the Nahavandi and Weinstein method.

Baclic [12] and Hill [13] both suggested employing independent functions which avoided the generation of such large matrix elements. Baclic put forward the use of $\phi_k(\xi) = \xi^k/k!$ while Hill introduced the use of the Chebyshev polynomials, that is with $\phi_k(\xi) = T_k(2\xi/\Lambda - 1)$. Hill indicated that large values of reduced length, Λ, did not cause the collocation method to break down with this use of the Chebyshev polynomials.† By way of example, he computed successfully the vector α for the symmetric case with $\Lambda = 100$ and $\Pi = 5$ using a polynomial of degree 20.

In this paper, we extend the work of Hill [13] to functions whose value always lies in the interval [0, 1] occupied by the temperature distributions $F(\xi)$ and $F'(\xi')$. In particular, we propose that the functions

$$\phi_k(\xi) = \frac{1}{2}\left\{T_k\left(\frac{2\xi}{\Lambda} - 1\right) + 1\right\} \tag{10}$$

be employed. Our results offered later are based on these functions.

4. CHOICE OF DATA POINTS

Parallels can be drawn between the method of collocation for the solution of integral equations and the method of *interpolation* for the approximation of one function by another. Without loss of generality, the symmetric case is considered and from equation (1) we define the function Ω by

$$\Omega(F(\xi)) = F(\Lambda - \xi) + e^{-\Pi}F(\xi) + \int_0^{\xi} K(\xi - \varepsilon)F(\varepsilon)\, d\varepsilon - 1. \tag{11}$$

† The Chebyshev polynomials $T_k(x)$ can be defined by the recurrence

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$.

If $F(\xi)$ were an exact solution of equation (1), then $\Omega(F(\xi)) = 0$ for $0 \leqslant \xi \leqslant \Lambda$. In the case, however, where $F(\xi)$ is an *estimate* of the exact solution, using the series expansion given by equation (6), we are left with a *residual* function $R(\xi)$ where

$$R(\xi) = \Omega\left(\sum_{j=0}^{n} \alpha_j \phi_j(\xi)\right). \tag{12}$$

Similarly, if we seek to represent a function $f(\xi)$ by an approximation $p_n(\xi)$ on [0, Λ] where $p_n(\xi)$ is a polynomial of degree n, then we can define an *error* function $e(\xi)$ equivalent to the residual function $R(\xi)$. Here

$$e(\xi) = f(\xi) - p_n(\xi).$$

In both interpolation and collocation, we select distinct data points $\{\xi_i | i = 0, 1, 2, \ldots, n\}$ and require that $e(\xi) = 0$ and $R(\xi) = 0$ at these $n+1$ positions. No attempt is made to minimise $|e(\xi)|$ or $|R(\xi)|$ at positions ξ on the interval [0, Λ] other that at $\{\xi_i | i = 0, 1, 2, \ldots, n\}$.

On the other hand, for interpolation, there is a well known theorem (see Morris [14], Atkinson [15] or Schwarz [16]) which specifies that the maximum value of $|e(\xi)|$ on [0, Λ] is minimized if we select the data points $\{\xi_i | i = 0, 1, 2, \ldots, n\}$ to be the zeros of the Chebyshev polynomial $T_{n+1}(x)$ mapped from the $[-1, 1]$ scale onto the [0, Λ] interval. These zeros take the form

$$\xi_i = \frac{\Lambda}{2}\left\{1 + \cos\left(\frac{2i+1}{n+1}\frac{\pi}{2}\right)\right\}$$

for $i = 0, 1, 2, \ldots, n$. The theorem relies upon the function $f(\xi)$ and the interpolating function $p_n(\xi)$ both being polynomials, f of degree $n+1$ and p_n of degree, at most n. The presence of the kernel $K(\xi - \varepsilon)$ in the integral equation precludes these assumptions. No guarantee can be given, therefore, that $|R(\xi)|$ will be minimised on [0, Λ] if the same zeros of $T_{n+1}(x)$ be used as data points in setting up the matrix equation (9) although some improvement over using equally spaced data points should be expected.

It turns out, however, that from physical considerations, it is useful to employ these *Chebyshev data points* since they are relatively closely spaced around the ends of the [0, Λ] interval which correspond to the entrances of the hot and cold gases to the regenerator. Hausen [17] pointed out that while the temperature distribution $F(\xi)$ was, in general, linear in the middle of the regenerator, most of the non-linearities are propagated from the regenerator entrances as a consequence of the inlet gas temperatures *not* varying with time in each period of regenerator operation. The result is that use of the Chebyshev data points enables these non-linearities to be more easily represented by the series expansion (6). Willmott and Duggan [11] explored this possibility and found that, indeed, modest but not dramatic economies could be made in the number of terms required in the series expansion (they

Table 1. Values of the thermal ratio, η_{REG}, and the number, N, of data points necessary to realise four-figure accuracy, as a function of reduced period, Π and reduced length, Λ. The values in the columns (a) were evaluated with *equally spaced* collocation points; those in the columns (b) relate to the *Chebyshev* collocation points. (Symmetric case.)

Π	0.1				1.0				5.0			
	(a)		(b)		(a)		(b)		(a)		(b)	
Λ	η_{REG}	N	η_{REG}	N	η_{REG}	N	η_{REG}	N	η_{REG}	N	η_{REG}	N
1	0.3332	3	0.3332	3	0.3221	4	0.3221	4	0.1877	3	0.1877	3
10	0.8333	4	0.8333	4	0.8322	10	0.8322	8	0.8086	8	0.8086	7
25	0.9259	8	0.9259	6	0.9257	14	0.9257	10	0.9214	12	0.9214	10
50	0.9615	8	0.9615	8	0.9614	22	0.9614	14	0.9604	22	0.9604	13
80	0.9756	10	0.9756	10	0.9753	28	0.9753	16	0.9751	30	0.9751	16

examined the Nahavandi and Weinstein [10] linearly independent functions) for an accurate solution of the integral equations, certainly as far as long regenerators are concerned, that is where $\Lambda/\Pi > 10$ and $\Lambda > 10$.

We have confirmed these observations and our results are shown in Table 1. It turns out, however, that for large reduced length, Λ, and *increasing* reduced period, Π, very significant economies can be effected using the Chebyshev data points. For example, with $\Lambda = 80$ and $\Pi = 5$, 30 equally spaced data points are required to achieve four-figure accuracy whereas only 16 Chebyshev collocation points are needed. This reflects the increased propagation of non-linear temperature behaviour from the regenerator entrances, consequent upon the constant inlet gas temperatures, as the period length, Π becomes larger. We would expect larger economies to be possible for even larger values of Π. The large values of reduced length, Λ, which we have been able to handle using the linearly independent functions defined by equation (10) is shown in this table. This matter is discussed later (see Table 2).

Table 2. Values of the thermal ratio, η_{REG}, and the number, N, of Chebyshev points of collocation necessary to achieve four-figure accuracy for values Λ and Λ/Π, where Λ is the reduced length and Π is the reduced period. (Symmetric case.)

Λ/Π	50		20		10	
Λ	η_{REG}	N	η_{REG}	N	η_{REG}	N
1.0	0.3333	3	0.3333	3	0.3332	2
3.0	0.6000	3	0.5999	3	0.5994	3
5.0	0.7142	3	0.7141	4	0.7134	4
7.0	0.7777	4	0.7775	3	0.7768	5
8.0	0.8000	4	0.7997	5	0.7989	6
10.0	0.8333	4	0.8330	6	0.8322	6
12.0	0.8571	4	0.8568	6	0.8559	7
16.0	0.8888	6	0.8886	7	0.8876	8
20.0	0.9090	6	0.9087	8	0.9078	8
25.0	0.9259	8	0.9256	8	0.9246	8
30.0	0.9374	8	0.9371	9	0.9362	8
40.0	0.9523	10	0.9520	10	0.9512	8
50.0	0.9614	10	0.9612	10	0.9604	10
100.0	0.9803	14	0.9802	11	0.9795	11
300.0	0.9935	17	0.9933	17	0.9930	18
500.0	0.9960	22	0.9958	21	0.9956	20
800.0	0.9974	21	0.9975	25	0.9974	27
1000.0	0.9980	28	0.9981	25	0.9973	20
2000.0	0.9989	33	0.9991	32	0.9991	32

5. BREAKDOWN OF THE METHOD OF ILIFFE [1]

Iliffe employed the quadrature method to solve equations (1)–(3). He approximated the integral by Simpson's rule. The temperatures $F(\xi)$ are computed, in his method, at positions $\{\xi_j = j\Delta\xi | j = 0, 1, 2, \ldots, N\}$ where $N\Delta\xi = \Lambda$. It is convenient to denote $F(\xi_j)$ by F_j while F'_j is defined in a similar manner for the cold period. It follows that

$$\int_0^{k\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon = \sum_{j=0}^{k} c_{j,k}K_{k-j}F_j \qquad (13)$$

where $K_{k-j} = K((k-j)\Delta\xi)$ and the $c_{j,k}$ are the coefficients in Simpson's rule. For k even, this takes the form

$$\int_0^{k\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon = \frac{\Delta\xi}{3}\{K_kF_0+4K_{k-1}F_1 + 2K_{k-2}F_2+\cdots+4K_1F_{k-1}+K_0F_k\}. \qquad (14)$$

For k odd, we split the integral

$$\int_0^{k\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon = \int_0^{3\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon + \int_{3\Delta\xi}^{k\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon \qquad (15)$$

and approximate the first integral by Simpson's three eighths rule and the second integral by Simpson's rule, as in equation (14). The three eighths rule takes the

form

$$\int_0^{3\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon = \frac{3\Delta\xi}{8}\{K_3F_0+3K_2F_1+3K_1F_2+K_0F_3\}.$$

For $k = 1$, we propose to use, here, the trapezium rule

$$\int_0^{\Delta\xi} K(\xi-\varepsilon)F(\varepsilon)\,\mathrm{d}\varepsilon = \frac{\Delta\xi}{2}\{K_1F_0+K_0F_1\} \quad (16)$$

whereas, more correctly, Iliffe estimated $F_{1/2}$ by interpolation using F_0, F_1, F_2, F_3 and then found the integral (16) by Simpson's rule again.

We construct the $(0 \ldots N)\times(0 \ldots N)$ matrices P, Q and R

$$P = \Delta\xi \begin{bmatrix} 0 & & & & & \\ K_1/2 & K_0/2 & & & & \\ K_2/3 & 4K_1/3 & K_0/3 & & & \\ 3K_3/8 & 9K_2/8 & 9K_1/8 & 3K_0/8 & & \\ K_4/3 & 4K_3/3 & 2K_2/3 & 4K_1/3 & K_0/3 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ K_N/3 & 4K_{N-1}/3 & 2K_{N-2}/3 & \ldots & \ldots & 4K_1/3 \quad K_0/3 \end{bmatrix} \quad (17)$$

where N is even, for example, $Q = \mathrm{e}^{-\Pi}I$ where I is the unit matrix and $R = [r_{i,j}]$ with $r_{j,N-j} = 1$ for $j = 0, 1, 2, \ldots, N$, and all other elements of R equal to zero. Upon application of equation (13) in equation (1) at positions $k = 0, 1, 2, \ldots, N$, the matrix equation (18) is formed.

$$(P+Q+R)\mathbf{F} = \mathbf{e} \quad (18)$$

Figure 1 shows the variation of the kernel function

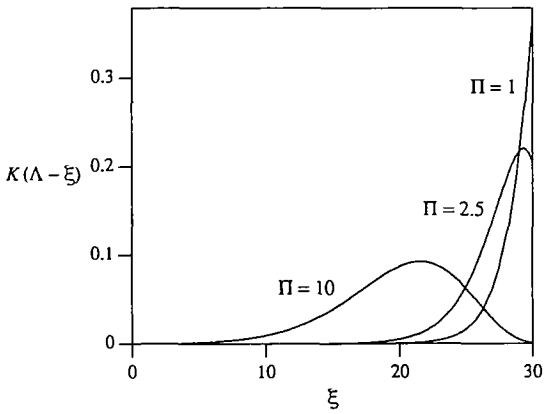

FIG. 1. Variation of the kernel $K(\Lambda-\xi)$ with ξ for $\Lambda = 30$.

with ξ and we see immediately that K_s is very small for large s, large Λ and small Π. As a consequence, in these circumstances, many effectively zero elements are introduced into the bottom rows of the matrix P. One way of looking at this is to say that, unless N is large, an insufficient number of large enough elements involving K_s with small s will be introduced into the bottom rows of the matrix P causing the matrix $P+Q+R$ to approach a singular state. This effect was observed by Willmott and Thomas [7] and was described in detail by them.

Another way of looking at the same problem is to say that it is difficult to evaluate the integral (13) for larger values of k, if the majority of the data points involve values of K_{k-j} which are very small and unrepresentative of $K(x)$ over the complete range of x under consideration.

This is the basic flaw in the method of Iliffe which makes its application difficult for large reduced length, Λ and small reduced period, Π (the so-called *long regenerator problem*), unless an unacceptably large number N of simultaneous equations (18) is used.

6. ROBUSTNESS OF THE SERIES EXPANSION METHODS

The weakness of the Iliffe method stems from the shape of the kernel, $K(x)$, for large Λ and small Π. This difficulty is overcome in the series expansion methods since the coefficients of the α_j in equation (8) involve *integrals* of $K(\xi_j-\varepsilon)\,\phi_j(\varepsilon)$ instead of the point values K_{k-j} used in the Iliffe method. Very small elements will not, therefore, be introduced into the relevant matrices. Baclic [12] offers analytical means of finding these integrals although they can also be found by Gaussian quadrature. It is this use of integrals in the construction of the coefficients of α_j in equation (8) and hence in the elements $a_{i,j}$ in equation (9) that provides the robustness of the series expansion method in this case. Even if the integrals are evaluated numerically, the more accurate methods of Gaussian quadrature can be employed. These are also more economical than the Newton–Cotes techniques which are unavoidable within the quadrature methods, such as that of Iliffe.

In this paper, we concentrate on the method of collocation to find the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. However, the same observations apply if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are determined by the Galerkin method, as described by Baclic [12]. The robustness of this method and that of the collocation method have the same origin. By way of a demonstration of this robustness, we have successfully computed the temperature performance of regenerators with reduced lengths in the range

$$1 \leqslant \Lambda \leqslant 2000$$

for Λ/Π equal to 50, 20 and 10. In doing so, we have used the linearly independent functions defined by equation (10) and the Chebyshev data points. These

results are shown in Table 2. Without loss of generality, these relate to the symmetric case.

The thermal regenerator problem is somewhat unusual, therefore, since in general, the Newton–Cotes methods are more often used in the solution of integral equations, since, as Baker [5] points out, they are generally easier to implement than an expansion method.

7. LIMITATION TO THE COLLOCATION METHOD

Although the *residual function* $R(\xi)$ is a non-linear function of the series expansion (6) it resembles the interpolation function which is a polynomial, in that it is $R(\xi) = 0$ at $\{\xi_i \mid i = 0, 1, 2, \ldots, n\}$. This means that using the Chebyshev expansion (10) can do little more than provide an *economization* or *telescoping* (see p. 425 of Atkinson [15]) of a solution provided by using $\phi_k(\xi) = \xi^k/k!$ This matter is also discussed by Fox and Parker [18].

It is for this reason that attention should be directed to the Galerkin method. It is suggested here that use of expansion (6) or simply $\phi_k(\xi) = T_k(2\xi/\Lambda - 1)$ is likely, hopefully, to yield a compact solution to equation (1) or equations (2) and (3), exploiting the sort of economy afforded by Chebyshev series in the least squares approximation of functions. Such a technique is described by Delves and Mohamed [6] and is called the *Fast Galerkin Algorithm*. The key problem is that in the Galerkin method, we seek to approximate $\Omega(F(\xi))$, not by a Chebyshev series but by a function Ω of a Chebyshev series, as in equation (12) with $\phi_i(\xi) = T_i(2\xi/\Lambda - 1)$. Choice of the relevant weighting function, however, enables the orthogonality of the Chebyshev functions to be exploited.

It is suggested that, as far as the collocation methods discussed in this paper are concerned, advantage can be taken of what economies are available by employing the Chebyshev functions defined by equation (6) and by using the Chebyshev data points in calculating the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

8. THE NON-SYMMETRIC CASE

The problems of economy in the series expansions and of stability discussed earlier apply to the symmetric and non-symmetric cases. No loss of generality has arisen, therefore, in discussing these matters in the context of equation (1). The observations and conclusions drawn apply equally to equations (2) and (3).

A computational problem arises in the non-symmetric case in addition, however. Solution of equation (9) involves the solution of $n+1$ simultaneous equations. The conventional approach to equations (2) and (3) involve the solution of $2n+2$ equations, where the solution vector is a concatination of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, defined by equations (6) and (7). Nahavandi and Weinstein [10] adapted such an approach for the collocation method as did Baclic and Dragutinovic [19] in their development of the Galerkin approach to these integral equations. This doubling of the number of equation results in the computational effort being multiplied by approximately eight. This problem can be resolved using the matrix method set out below.

9. THE MATRIX METHOD

Equation (2) can be re-written for the cold period in the pth cycle

$$F'^{(p)}(\xi') = e^{-\Pi'} F^{(p)}(\xi') + \int_{\xi'}^{\Lambda'} K'(\varepsilon - \xi') F^{(p)}(\varepsilon)\, d\varepsilon. \tag{19}$$

Inserting expansions (6) and (7) and denoting the vectors for the pth cycle by $\boldsymbol{\alpha}^{(p)}$ and $\boldsymbol{\beta}^{(p)}$ yields

$$\sum_{j=0}^{n} \beta_j^{(p)} \phi_j(\xi') = \sum_{j=0}^{n} \alpha_j^{(p)} \left\{ e^{-\Pi'} \phi_j(\xi') + \int_{\xi'}^{\Lambda'} K'(\varepsilon - \xi') \phi_j(\varepsilon)\, d\varepsilon \right\}. \tag{20}$$

Application of equation (20) at distinct positions $\{\xi_i' \mid i = 0, 1, 2, \ldots, n\}$ yields a matrix equation, namely

$$\Phi' \boldsymbol{\beta}^{(p)} = \Gamma' \boldsymbol{\alpha}^{(p)} \tag{21}$$

where Φ' and Γ' are $(0 \ldots n) \times (0 \ldots n)$ matrices. These are defined by $\Phi' = [\phi_j(\xi_i')]$ and

$$\Gamma' = \left[e^{-\Pi'} \phi_j(\xi_i') + \int_{\xi_i'}^{\Lambda'} K'(\varepsilon - \xi_i') \phi_j(\varepsilon)\, d\varepsilon \right].$$

The vector $\boldsymbol{\beta}^{(p)}$ can be computed from $\boldsymbol{\alpha}^{(p)}$ using

$$\boldsymbol{\beta}^{(p)} = \Phi'^{-1} \Gamma' \boldsymbol{\alpha}^{(p)}. \tag{22}$$

Equation (3) for the hot period in the pth cycle is developed in a similar manner. It is somewhat rearranged to yield

$$F^{(p+1)}(\xi) = e^{-\Pi} F^{(p)}(\xi) + \int_0^{\xi} K(\xi - \varepsilon) F^{(p)}(\varepsilon)\, d\varepsilon + \left\{ 1 - e^{-\Pi} - \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\}. \tag{23}$$

Substituting again expansions (6) and (7), this time into equation (23) generates

$$\sum_{j=0}^{n} \alpha_j^{(p+1)} \phi_j(\xi) = \sum_{j=0}^{n} \beta_j^{(p)} \left\{ e^{-\Pi} \phi_j(\xi) + \int_0^{\xi} K(\xi - \varepsilon) \phi_j(\varepsilon)\, d\varepsilon \right\} + \left\{ 1 - e^{-\Pi} - \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\}. \tag{24}$$

Application of equation (24) at distinct data points

$\{\xi_i \mid i = 0, 1, 2, \ldots, n\}$ produces an equation in matrix form which is

$$\Phi\boldsymbol{\alpha}^{(p+1)} = \Gamma\boldsymbol{\beta}^{(p)} + \boldsymbol{\zeta} \tag{25}$$

where $\Phi = [\phi_j(\xi_i)]$ and

$$\Gamma = \left[e^{-\Pi}\phi_j(\xi_i) + \int_0^{\xi_i} K(\xi_i - \varepsilon)\phi_j(\varepsilon)\,d\varepsilon \right].$$

The vector $\boldsymbol{\zeta} = [\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n]^T$ is defined by

$$\zeta_i = \left\{ 1 - e^{-\Pi} - \int_0^{\xi_i} K(\xi_i - \varepsilon)\,d\varepsilon \right\}. \tag{26}$$

Equation (25) can now be re-arranged to give

$$\boldsymbol{\alpha}^{(p+1)} = \Phi^{-1}\Gamma\boldsymbol{\beta}^{(p)} + \Phi^{-1}\boldsymbol{\zeta}. \tag{27}$$

Equation (28) is obtained by the substitution for $\boldsymbol{\beta}^{(p)}$ from equation (22); it takes the form

$$\boldsymbol{\alpha}^{(p+1)} = \Phi^{-1}\Gamma\Phi'^{-1}\Gamma'\boldsymbol{\alpha}^{(p)} + \Phi^{-1}\boldsymbol{\zeta}. \tag{28}$$

If $\Phi^{-1}\Gamma\Phi'^{-1}\Gamma'$ is denoted by M and $\Phi^{-1}\boldsymbol{\zeta}$ by $\boldsymbol{v}$, then equation (28) becomes

$$\boldsymbol{\alpha}^{(p+1)} = M\boldsymbol{\alpha}^{(p)} + \boldsymbol{v}. \tag{29}$$

At cyclic equilibrium, $\boldsymbol{\alpha}^{(p)} = \boldsymbol{\alpha}^{(p+1)} = \boldsymbol{\alpha}$ in which case, equation (29) takes the form

$$(I - M)\boldsymbol{\alpha} = \boldsymbol{v}$$

and the vector $\boldsymbol{\alpha}$ located by solving the set of $n+1$ linear equations, namely

$$\boldsymbol{\alpha} = (I - M)^{-1}\boldsymbol{v}. \tag{30}$$

Once $\boldsymbol{\alpha}$ has been found, the vector $\boldsymbol{\beta}$ is obtained from

$$\boldsymbol{\beta} = \Phi'^{-1}\Gamma'\boldsymbol{\alpha}. \tag{31}$$

Equation (30) involves the solution of only $n+1$ instead of $2n+2$ equations. The calculation of the vector $\boldsymbol{\beta}$ requires only a matrix multiplication. It is necessary, however, to find the matrices $\Phi^{-1}\Gamma$ and $\Phi'^{-1}\Gamma'$ together with the vector $\Phi^{-1}\boldsymbol{\zeta}$. This can be realised economically by performing single LU decompositions of Φ and Φ', each equivalent in computational effort, to solving $n+1$ linear equations. The matrices $\Phi^{-1}\Gamma$ and $\Phi'^{-1}\Gamma'$ and the vector $\Phi^{-1}\boldsymbol{\zeta}$ are then found by the conventional forward and back substitution processes. A further economy can be made if the same data points ξ_i are used for the collocation scheme in the hot and cold periods, in which case $\Phi' = \Phi$.

The procedure described above represents a development of the matrix approach to regenerator simulations suggested recently by Willmott *et al.* [20] applied initially to the Iliffe method for solving equations (2) and (3). The same economies can be realised for the method of Galerkin. In this case, the matrices Φ' and Γ' for equation (21) take the forms

$$\Phi' = \left[\int_0^{\Lambda'} \phi_j(\xi')\phi_i(\xi')\,d\xi' \right] \tag{32}$$

$$\Gamma' = \left[\int_0^{\Lambda'} \left\{ e^{-\Pi}\phi_j(\xi') + \int_{\xi'}^{\Lambda'} F(\varepsilon - \xi')\phi_j(\varepsilon)\,d\varepsilon \right\} \phi_i(\xi')\,d\xi' \right]. \tag{33}$$

The matrix Φ' assumes diagonal form if the functions $\phi_j(\xi')$ are orthogonal. The matrices Φ and Γ, and the vector $\boldsymbol{\zeta}$ for the hot period can be developed in a similar manner. This approach extends the work of Baclic [12] in which the Galerkin method for equation (1) is described, together with the later work of Baclic and Dragutinovic [19] which encompasses the non-symmetric problem. Details of this development and the application of the Fast Galerkin Algorithm will be described in a future paper.

10. CONCLUDING REMARKS

The series expansion methods for the Nusselt integral equations (1)–(3) are shown to be robust. They should be used in preference to the computationally more economical Iliffe quadrature scheme if it is at all likely that software implementing a method of solution of the integral equations will be called upon to handle cases of the *long regenerator* problem. It is suggested that the linearly independent functions defined by equation (10) be employed since the function values lie on the same [0, 1] range occupied by the dimensionless temperatures $F(\xi)$ and $F'(\xi')$.

Advantage should be taken of the matrix scheme described here for the non-symmetric case since the number of equations required to be solved is halved. For collocation schemes, the Chebyshev data points should be employed for both the symmetric and non-symmetric cases. The same Chebyshev data points should be used for the hot and cold periods for the non-symmetric case so that computational advantage can be taken of the relation $\Phi' = \Phi$.

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